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Fake Rings, Fake Modules, and Duality

D. D. ANDERSON*

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

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A duality theory for modules over a commutative ring is developed using lattice modules. Using this duality theory several classical results for Noetherian modules are dualized to Artinian modules. This theory of duality also applies to certain other algebraic systems.

1. INTRODUCTION

Throughout this paper all rings are commutative with identity. Noetherian modules have been extensively studied, while Artinian modules have received much less attention (see, for example, Matlis [10, 11] and Kirby [9].) The ascending chain condition (ACC) and the descending chain condition (DCC) being dual notions, it is reasonable to believe that there should be some duality between Noetherian modules and Artinian modules. Matlis [10] has developed such a duality theory over a complete local ring. The purpose of this paper is to develop a duality theory for modules over an arbitrary commutative ring. Our duality theory is based on the concept of a fake module or lattice module (see Section 2). While not every module has a dual module, every module does have a natural dual fake module. By proving theorems about modules for the larger class of fake modules, we are able to dualize the results.

In Section 2 we introduce fake modules and give our basic duality construction. We also consider principal-like elements acting on the fake module. In Section 3 we dualize the theory of primary decomposition to coprimary decomposition. We prove the usual existence and uniqueness theorems. The theory of R-sequences and co-R-sequences is developed in Section 4. In Section 5 we study fake modules over fake rings (i.e., multiplicative lattices). We show how to generalize the results of Sections 1–4 to fake rings. Thus our results hold for other algebraic systems such as graded rings or semigroups.

We make no claim to an exhaustive application of this duality theory, but, rather give several different applications. Another method of duality, using

* Present address: University of Iowa, Iowa City, Iowa 52242.

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Abelian categories, was introduced by Moore [12, 13]. We believe that the fake module formulation has several advantages. For example, it applies to certain algebraic systems such as semigroups, whose "modules" do not form an Abelian category (see Section 5).

2. Fake Modules

Let R be a ring and N be an R-module. We use $L(R)$ and $L(N)$ to denote the lattice of ideals of R and the lattice of R-submodules of N, respectively. The action of R on N induces a natural action of $L(R)$ on $L(N)$ satisfying the following conditions:

(1) $(\sum J\alpha)A = \sum (J\alpha A)$,

(2) $J(\sum A\alpha) = \sum JA\alpha$,

(3) $(JK)A = J(KA)$,

(4) $RA = A$, and

(5) $0A = 0_N$,

for $J\alpha$, J, $K \in L(R)$ and $A\alpha$, $A \in L(N)$. Here 0_N denotes the zero submodule of N.

We define a *fake R-module* to be a complete modular lattice N together with a mapping $L(R) \times N \to N$ satisfying the above five conditions (where now the sum $\sum$ in $L(N)$ is replaced by the join $\vee$ and 0_N is now the least element of N). Let N and N' be two fake R-modules. A map $\theta\colon N \to N'$ is called an *R-homomorphism* if it preserves arbitrary joins and satisfies $\theta(JA) = J\theta(A)$ for all ideals J of R and all $A \in N$. The map θ is called an *R-isomorphism* if it is bijective. Thus an R-isomorphism preserves arbitrary joins and meets. We note that for any R-module N, $L(N)$ is a fake R-module. Fake modules have also been called lattice modules and have been studied elsewhere (see Section 5).

We now give our fundamental construction. Let N be a fake R-module and let N^* be the lattice dual of N, that is, N with the partial order reversed. Thus $(N^*, \leqslant^*)$ is a complete lattice with join $\vee^* A\alpha = \wedge A\alpha$, meet $\wedge^* A\alpha = \vee A\alpha$ (for all subsets $\{A\alpha\} \subseteq N$), $0_{N^*} = I_N$, and $I_{N^*} = 0_N$ (we always use 0_N (I_N) to denote the least (greatest) element for a complete lattice N). The interesting property of N^* is that it is still a fake R-module under a new scalar product given by residuation.

THEOREM 2.1. *Let N be a fake R-module and let N^* be the lattice dual of N. Then N^* is a fake R-module with respect to the new scalar product $J * A = (A : J) = \vee \{X \in N \mid JX \leqslant A\}$.*

Proof. We verify that N^* satisfies the five necessary properties. For $J\alpha$, $J, K \in L(R)$ and $A\alpha$, $A \in N$ we have

(1) $(\sum J\alpha) * A = \wedge (A : J\alpha) = \vee^* J\alpha * A$,

(2) $J * (\vee^* A\alpha) = (\wedge A\alpha : J) = \wedge (A\alpha : J) = \wedge (J * A\alpha) = \vee^* J * A\alpha$,

(3) $(JK) * A = (A : JK) = (A : KJ) = (A : K) : J = J * (K * A)$,

(4) $R * A = (A : R) = A$, and

(5) $0 * A = (A : 0) = I_N = 0_{N^*}$.

Let N be a fake R-module. To get interesting results it is necessary to assume that the principal ideals of R act nicely on the fake R-module N. An ideal M of R is called an *N-meet (-join) principal* if $M(A \wedge (B : M)) = MA \wedge B$ $((A \vee MB : M) = (A : M) \vee B)$ for all $A, B \in N$. We call M *N-principal* if it is both N-meet and N-join principal. A fake R-module N is said to be *meet-principally, join-principally,* or *principally generated* if every ideal of R is a sum of N-meet-principal, N-join-principal, or N-principal ideals, respectively. If N is a (real) R-module, then any principal ideal in R is N-principal. Thus an R-module is principally generated. For fake R-modules, meet-principal ideals and join-principal ideals are dual notions.

THEOREM 2.2. *Let N be a fake R-module. Then an ideal M of R is N-meet (-join) principal if and only if M is N^*-join (-meet) principal. In particular, M is N-principal if and only if it is N^*-principal.*

Proof. In N^*, we denote residuation by $\overset{*}{*}$. Then for $A \in N$ and $J \in L(R)$ we have $(A \overset{*}{*} J) = \vee^*\{B \in N^* \mid J * B \leqslant {}^*A\} = \wedge \{B \in N \mid (B : J) \geqslant A\} = JA$. The following identity shows that M is N-meet principal implies that M is N^*-join principal:

$$((A \vee^* M * B) \overset{*}{*} M) = (A \wedge (B : M) \overset{*}{*} M) = M(A \wedge (B : M))$$
$$= MA \wedge B = (A \overset{*}{*} M) \vee^* B.$$

The other implications are similar.

Let N be an R-module. Then $L(N)$ is a $L(N)$-principally generated fake R-module and hence $L(N)^*$ is a $L(N)^*$-principally generated fake R-module. We say that an R-module N has a *dual* if there exists an R-module M such that $L(N)^*$ and $L(M)$ are R-isomorphic. While an R-module need not have a dual module, it always has a fake dual module (for a discussion of the existence of dual modules see [2]).

Suppose that N is an Artinian R-module, then $L(N)$ is an Artinian $L(N)$-principally generated fake R-module. Here is our method of duality. Suppose that we have a theorem which is true for Noetherian R-modules. If we can prove this theorem for the larger class of Noetherian principally generated fake R-modules, then the dual of the theorem will be true for the class of Artinian principally generated fake R-modules and hence in particular for Artinian R-modules.

3. Primary Decomposition

We show that the primary decomposition theory and its dual hold for suitably defined fake R-modules.

Let N be a fake R-module. For $B \in N$ we define $\mathrm{rad}(B) = \{x \in R \mid (x)^n I_N \leqslant B,$ n a positive integer$\}$. It is clear that $\mathrm{rad}(B)$ is an ideal of R. We say that $B \in N$ is *primary* if (1) $B \neq I_N$ and (2) $JA \leqslant B$ implies $A \leqslant B$ or $J \leqslant \mathrm{rad}(B)$. If N satisfies ACC, then $\mathrm{rad}(B)^n I_N \leqslant B$ for some positive integer n. We say that $A \in N$ has a *strong normal decomposition* if $A = Q_1 \wedge \cdots \wedge Q_n$ where the Q_i's are primary, the intersection is irredundant, the $P_i = \mathrm{rad}(Q_i)$ are distinct primes, and $P_i{}^s I_N \leqslant Q_i$ for some positive integer s. Finally we say that N satisfies the *strong normal decomposition theory* if every element $\neq I_N$ has a strong normal decomposition.

A slight modification of [4, Theorem 3.1] yields the following decomposition theorem.

Theorem 3.1. *Let R be a commutative ring and N be a fake R-module. Suppose that N satisfies ACC and is meet-principally generated. Then any meet-irreducible element of N is primary and hence N satisfies the strong normal decomposition theory.*

We also have the usual uniqueness theorem.

Theorem 3.2. *Let R be a commutative ring and N be a fake R-module. Suppose that $Q_1 \wedge \cdots \wedge Q_n$ and $Q_1' \wedge \cdots \wedge Q_{n'}'$ are two strong normal decompositions of $A \in N$ and* $\mathrm{rad}(Q_i) = P_i$ $(\mathrm{rad}(Q_i') = P_i')$. *Let* $\mathrm{Ass}(N/A) = \{P$ *a prime ideal of* $R \mid \exists C \in N \ni P = (A : C)\}$. *Then,* $Z(N/A) = \{r \in R \mid A : (r) \neq A\} = \bigcup \{P \in \mathrm{Ass}(N/A)\}$ *and* $\{P_1, \ldots, P_n\} = \mathrm{Ass}(N/A) = \{P_i', \ldots, P_{n'}'\}$.

Now we dualize our results on primary decomposition to coprimary decomposition. We review what is meant by coprimary decomposition. Let R be a commutative ring and A be an R-module. A nonzero submodule B of A is *coprimary* if for $r \in R$ and any submodule L of A, $rB \leqslant L$ implies $B \leqslant L$ or $r^n B = 0$ for some n. It is easy to verify that $(0 : B)$ is primary and $\mathrm{rad}(0 : B)$ is prime.

Let R be a commutative ring and N be a fake R-module. An element $B \in N$ is *coprimary* if (1) $B \neq 0_N$ and (2) $JB \leqslant L$ implies $J \leqslant \mathrm{rad}(0 : B)$ or $B \leqslant L$. We say that $A \in N$ has a *strong normal codecomposition* if $A = Q_1 \vee \cdots \vee Q_n$, where the Q_i's are coprimary, the join is irredundant, the $P_i = \mathrm{rad}(0 : Q_i)$ are distinct primes, and $P_i{}^s Q_i = 0_N$ for some integer s.

The duality between primary and coprimary elements is established by the following lemma, the proof of which is straightforward and hence omitted.

Lemma 3.3. *Let R be a commutative ring and N be a fake R-module. For $B \in N$ we have*

(1) $\operatorname{rad}(B) = \operatorname{rad}(0_{N^*} \overset{*}{*} B)$,

(2) $\operatorname{rad}(B)^n I_N \leqslant B$ *if and only if* $\operatorname{rad}(0_{N^*} \overset{*}{*} B)^n * B = 0_{N^*}$,

(3) *$B \in N$ is primary if and only if $B \in N^*$ is coprimary,*

(4) *if N satisfies DCC, then B coprimary implies that* $\operatorname{rad}(0_N : B)$ *is prime and* $\operatorname{rad}(0_N : B)^n \cdot B = 0_N$ *for some positive integer n.*

Applying our method of duality to Theorem 3.1 and 3.2 yields the dual theorems for coprimary decomposition.

THEOREM 3.1*. *Let R be a commutative ring and N be a fake R-module satisfying DCC. Further assume that N is join-principally generated. Then any join-irreducible element of N is coprimary and hence N satisfies the strong normal codecomposition theory.*

THEOREM 3.2*. *Let R be a commutative ring and N be a fake R-module. Suppose that $Q_1 \vee \cdots \vee Q_n$ and $Q_1' \vee \cdots \vee Q'_{n'}$ are two strong normal codecompositions of $A \in N$ and that* $\operatorname{rad}(0 : Q_i) = P_i$ *and* $\operatorname{rad}(0 : Q_i') = P_i'$. *Let* $\text{co-Ass}(N/A) = \{P$ *a prime ideal in* $R \mid \exists C \in N \ni P = (C : A)\}$. *Then $n = n'$ and* $\{P_1, \ldots, P_n\} = \text{co-Ass}(N/A) = \{P_1', \ldots, P'_{n'}\}$. *Furthermore,* $0(N/A) = \{r \in R \mid (r)A \neq A\} = \bigcup \{P \in \text{co-Ass}(N/A)\}$.

As an immediate corollary of the previous theorems we get the coprimary decomposition theorem for Artinian modules.

COROLLARY. *Let R be a ring and N be an Artinian R-module. Then every nonzero submodule A of N is a finite sum of coprimary submodules. Any two strong normal codecompositions have the same length and same set of coassociated primes. In fact,* $\text{co-Ass}(N/A) = \{P$ *a prime ideal in* $R \mid P = (c : A)$ *for some* $C \in N.\}$ *and* $\mathcal{O}(N/A) = \{r \in R \mid rA \neq A\} = \bigcup \{P \mid P \in \text{co-Ass}(N/A)\}$.

As an application of the primary decomposition theory, we prove the Krull intersection theorem for fake R-modules.

THEOREM 3.4. *Let N be a fake R-module which satisfies the strong normal decomposition theory. Then for any ideal J of R and any $M \in N$ we have* $\bigwedge_{n=1}^{\infty} J^n M = J(\bigwedge_{n=1}^{\infty} J^n M)$.

Proof. Let $J(\bigwedge_{n=1}^{\infty} J^n M) = Q_1 \wedge \cdots \wedge Q_s$, where Q_i is P_i-primary. Then $J(\bigwedge_{n=1}^{\infty} J^n M) \leqslant Q_i$ implies $\bigwedge_{n=1}^{\infty} J^n M \leqslant Q_i$ or $J \leqslant P_i$, in which case $J^n M \leqslant J^n I_N \leqslant Q_i$ for n large. In either case $\bigwedge_{n=1}^{\infty} J^n M \leqslant Q_i$ and hence, $J(\bigwedge_{n=1}^{\infty} J^n M) = \bigwedge_{n=1}^{\infty} J^n M$.

THEOREM 3.4*. *Let N be a fake R-module satisfying the strong normal codecomposition theory. For any ideal J of R and $M \in N$ we have* $\bigvee_{n=1}^{\infty} (M : J^n) = (\bigvee_{n=1}^{\infty} (M : J^n) : J)$.

As an immediate corollary we have

COROLLARY. *Let R be a ring and N an Artinian R-module. Let J be an ideal of R and M be a submodule of N, then* $\bigcup_{n=1}^{\infty}(M : J^n) = (\bigcup_{n=1}^{\infty}(M : J^n) : J)$.

If we allow the ring R to be noncommutative, we may still discuss fake R-modules. Now, however, the dual of a left fake R-module is a right fake R-module. The notions of primary decomposition and tertiary decomposition may be extended to fake R-modules and dualized to coprimary decomposition and cotertiary decomposition.

4. R-SEQUENCES

We next study the theory of grade and cograde for fake R-modules. For the theory of grade in the R-module case, see [8]. Co-R-sequences and cograde were introduced in [11] for the case of Artinian modules over a Noetherian ring.

DEFINITION. Let R be a commutative ring and N be a fake R-module. A sequence $X_1, \ldots, X_n$ of N-principal ideals of R is called an *R-sequence on N* if

(1) $(X_1 + \cdots + X_n)I_N \neq I_N$, and

(2) $(0_N : X_1) = 0_N$ and $((X_1 + \cdots + X_{i-1})I_N : X_i) = (X_1 + \cdots + X_{i-1})I_N$ for $i = 2, \ldots, n$. We call $X_1, \ldots, X_n$ a *co-R-sequence on* N if $X_1, \ldots, X_n$ is an R-sequence on N^*.

Thus $X_1, \ldots, X_n$ is a co-R-sequence on N if and only if

(1) $(0_N : X_1 + \cdots + X_n) \neq 0_N$ and

(2) $X_1 I_N = I_N$ and $X_i(0_N : X_1 + \cdots + X_{i-1}) = (0_N : X_1 + \cdots + X_{i-1})$ for $i = 2, \ldots, n$.

Suppose that $X_1, \ldots, X_n$ is an R-sequence on N. Then $0 < X_1 I_N < (X_1 + X_2)I_N < \cdots < (X_1 + \cdots + X_n)I_N$. For if $(X_1 + \cdots + X_i)I_N = (X_1 + \cdots + X_{i-1})I_N$, then $(X_1 + \cdots + X_{i-1})I_N = ((X_1 + \cdots + X_{i-1})I_N : X_i) = ((X_1 + \cdots + X_i)I_N : X_i) = I_N$, which contradicts (1) in the definition of an R-sequence. Thus if N satisfies ACC (DCC), maximal (co-) R-sequences on N do exist. The next theorem and its dual show that any two such maximal (co-) R-sequences have the same length.

THEOREM 4.1. *Let R be a commutative ring, J be an ideal in R, and N be a principally generated fake R-module satisfying ACC. Further suppose that $JI_N \neq I_N$. Then any two maximal R-sequences on N contained in J have the same length.*

Proof. A slight modification of [8, Theorem 121] yields the desired result.

Dualizing we have

THEOREM 4.1*. *Let R be a commutative ring, J be an ideal in R, and N be a principally generated fake R-module satisfying DCC. Further assume that $(0_N : J) \neq 0_N$. Then any two maximal* co-R-*sequences on N contained in J have the same length.*

COROLLARY. *Let R be a commutative ring (not necessarily Noetherian), J an ideal in R, and N an Artinian R-module. Assume that $(0_N : J) \neq 0_N$. Then any two maximal* co-R-*sequences on N contained in J have the same length.*

5. FAKE RINGS

The concept of the lattice of ideals of a commutative ring is one particular example of a multiplicative lattice. (A multiplicative lattice L is a complete lattice L on which has been defined a commutative, associative product which has I_L as a multiplicative identity and in which multiplication distributes over arbitrary joins. See [1, 3], or [4].) For a multiplicative lattice L, we define a fake L-module N to be a complete lattice N satisfying

(1) $(\vee J\alpha)A = \vee J\alpha A$,

(2) $J(\vee A\alpha) = \vee JA\alpha$,

(3) $(JK)A = J(KA)$,

(4) $I_L A = A$, and

(5) $0_L A = 0_N$,

for all J, $J\alpha$, $K \in L$ and A, $A\alpha \in N$.

Our duality construction given in Section 2 carries over to fake L-modules (where L is now a multiplicative lattice). The notion of principal elements also extends to fake L-modules. Fake L-modules have been studied elsewhere under the name lattice module, see [5–7, 14]. The results of Sections 2 and 3 carry over to L-modules without any difficulty. However, the results of Section 4 require that the multiplicative lattice satisfy the lattice analog of the following well-known result: If an ideal I of a commutative ring is contained in a finite union of prime ideals, then it is contained in one of them. The lattice analog of this theorem is the so-called *union condition on primes*: If L is a multiplicative lattice and N is a fake L-module; $A \in L$ and $P_1, \ldots, P_n$ prime elements of L with $A \nleq P_1, \ldots, P_n$, then there exists an N-principal element $E \leqslant A$ with $E \nleq P_1, \ldots, P_n$.

We conclude this paper by giving some examples of fake modules over fake rings. For any R-module N, $L(N)$ is of course an $L(R)$-module. More generally, let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be a graded ring and N be a graded R-module. Then $L_G(R)$, the lattice of graded ideals of R, is a multiplicative lattice, and $L_G(N)$, the lattice of graded R-submodules of N, is a $L_G(R)$-module. Thus the results of Section 3 yield the usual homogeneous primary and coprimary decomposition theories for graded submodules. The results of Section 4 do not carry over to graded rings because $L_G(R)$ need not satisfy the union condition on primes, i.e., if a homogeneous ideal I is not contained in a union $P_1 \cup \cdots \cup P_n$ of homogeneous prime ideals, it does not follow that there exists a homogeneous element of I not contained in this union. There is, however, one particular instance where R does satisfy this condition, namely, where R_0 is Artinian.

For our final example, let S be a commutative monoid with zero and let T be a set with a distinguished element $*$. Suppose that S acts on T, that is, there exists a mapping $S \times T \to T$ satisfying $(s_1 s_2)t = s_1(s_2 t)$, $1t = t$, and $0t = *$ for all $s_1, s_2 \in S$ and $t \in T$. $L(S)$, lattice of ideals of S, is a quasilocal multiplicative lattice [3], and $L(T)$, the lattice of S-invariant subsets of T, is easily seen to be an $L(S)$-module.

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